Ch2 : Determinants.

I) Determinants by cofactor Expansion.

→ Minor of entry a_{ij} (M_{ij}) :

Determinants of the submatrix that remains after deleting the ith row and jth columns

هو det لجزء من ال matrix نحصل على هذا الجزء من خلال حذف العمود و الصف المذكوران

→ Cofactor of entry a_{ij} (C_{ij})

$$C_{ij} = (-1)^{i+j} M_{ij} = \pm M_{ij}$$

Note !!
M_{ij} , C_{ij} only for square matrices

→ Determinant of 2 x 2 matrix

if A is a 2 x 2 matrix :

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{then } \det(A) = ad - bc.$$

Example :

let $A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 5 & 2 \\ -4 & 6 & 1 \end{bmatrix}$, Find M_{21} , M_{22} , M_{32} and C_{21} , C_{22} , C_{32}

$$M_{21} = \begin{vmatrix} 3 & -1 \\ 6 & 1 \end{vmatrix} = 3 + 6 = 9 \rightarrow C_{21} = (-1)^3 \, 9 = -9$$

$$M_{22} = \begin{vmatrix} 2 & -1 \\ -4 & 1 \end{vmatrix} = 2 - 4 = -2 \rightarrow C_{22} = (-1)^4 \, -2 = -2$$

$$M_{32} = \begin{vmatrix} 2 & -1 \\ 4 & 2 \end{vmatrix} = 4 + 4 = 8 \rightarrow C_{32} = (-1)^5 \, + 8 = -8$$

→ Determinant of $n \times n$ matrix

Example 1:

let $A = \begin{bmatrix} 2 & 4 & 3 \\ -5 & 6 & 1 \\ 3 & 4 & 2 \end{bmatrix}$, Find det (A)

Note !!

ممكن أن نجد الـ det عن

طريق الصفوف أو الأعمدة

1. By row 1 , i = 1

$$\det(A) = \underset{a_{11}}{2} \underset{C_{11}(+)}{\begin{vmatrix} 6 & 1 \\ 4 & 2 \end{vmatrix}} - \underset{a_{12}}{4} \underset{C_{12}(-)}{\begin{vmatrix} -5 & 1 \\ 3 & 2 \end{vmatrix}} + \underset{a_{13}}{3} \underset{C_{13}(+)}{\begin{vmatrix} -5 & 6 \\ 3 & 4 \end{vmatrix}}$$

$$= 2\,(8) - 4\,(-13) + 3\,(-38) = -46$$

2. By row 2 , i = 2

$$\det(A) = 5 \begin{vmatrix} 4 & 3 \\ 4 & 2 \end{vmatrix} + 6 \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} - 1 \begin{vmatrix} 2 & 4 \\ 3 & 4 \end{vmatrix}$$

$$= 5\,(-4) + 6\,(-5) - 1\,(-4) = -46$$

3. By column 1 ; j = 1

$$\det(A) = 2 \begin{vmatrix} 6 & 1 \\ 4 & 2 \end{vmatrix} + 5 \begin{vmatrix} 4 & 3 \\ 4 & 2 \end{vmatrix} + 3 \begin{vmatrix} 4 & 3 \\ 6 & 1 \end{vmatrix}$$

$$= 2\,(8) + 5\,(-4) + 3\,(-14) = -46$$

Note !!

الـ det دائماً يأخذ نفس القيمة لكل

matrix مهما كانت الطريقة التي

إستخدمناها لإيجاده

على سبيل المثال في الـ ex أعلاه

الـ det دائماً تساوي (-46)

Example 2

let $A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 1 \end{bmatrix}$ find det (A)

1) $i = 1$

$$\det(A) = 2 \begin{vmatrix} 6 & 7 \\ 9 & 1 \end{vmatrix} - 3 \begin{vmatrix} 5 & 7 \\ 8 & 1 \end{vmatrix} + 4 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix}$$

$$= 2(-57) - 3(-51) + 4(-3)$$

$$= 27$$

2) $i = 2$

$$\det(A) = -5 \begin{vmatrix} 3 & 4 \\ 9 & 1 \end{vmatrix} + 6 \begin{vmatrix} 2 & 4 \\ 8 & 1 \end{vmatrix} - 7 \begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix}$$

$$= -5(-33) + 6(-30) - 7(-6)$$

$$= 27$$

→ Theorm:

If A is $n \times n$ diagonal or upper triangular or lower triangular Matrix then:

$$\det(A) = a_{11} \times a_{22} \times a_{33} \dots \times a_{nn}$$

Example:

Let: $A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}}_{\text{diagonal matrix}}$ $B = \underbrace{\begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 5 \\ 0 & 0 & 8 \end{bmatrix}}_{\text{upper triangle matrix}}$ $C = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 5 & 2 & 0 \\ 4 & 6 & 3 \end{bmatrix}}_{\text{lower triangle matrix}}$ then:

$\det(A) = 1 \times 2 \times 3$

$\det(B) = 1 \times 2 \times 8$

$\det(C) = 1 \times 2 \times 3$

→ Properties of determinants:

1. Let A a square Matrix, if A has a row of zeros or column of zeros, then $\det(A) = 0$
2. Let A a square Matrix, $\det(A) = \det(A^T)$
3. Let A and B two n×n matrices then,
   $\det(A \times B) = \det(A) \times \det(B)$
4. Row Operations:
   The row operation consist of the following:
   a) Switch two rows
   b) Multiply a row by a non zero nb
   c) Replace a row by a multiple of another row added to itself.

↳ Examples:

a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{array}{l} R_1 \\ R_2 \end{array} \Rightarrow \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \begin{array}{l} R_2 \\ R_1 \end{array}$

b/c) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 2 & 3 \end{bmatrix}$ $R_2 - 2R_1 \to R_2$

→ Theorems:

1. Switch rows or columns:
   If matrix B result from switching two rows or columns of the matrix A, then:
   $\det(B) = -\det(A)$

↳ Ex: $A \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$

$\det(A) = -2$ $\det(B) = +2$

N.B !!
التبديل الواحد عبارة عن (-)

2. Multiplying a row by a scalar

If B a Matrix results from a Matrix A by multiply one row or one column of A by a scalar K, then $\det(B) = K \det(A)$

إذا matrix معيّنة ضربت صف واحد أو عمود واحد فقط
برقم ثابت، الـ matrix الجديدة الناتجة عن هذا الضرب تتبع
القاعدة هذه.

↳ Ex:

$$A = \begin{bmatrix} 2 & 5 \\ 4 & -3 \end{bmatrix} \xrightarrow{\times 3} \quad , \quad B = \begin{bmatrix} 2 & 5 \\ 12 & -9 \end{bmatrix}$$

$\det(A) = -26$

$$\det(B) = 3\det(A) = 3(-26) = -78$$

3. Multiplying a matrix by a scalar

if $B = KA$, then $\det(B) = K^n \det(A)$ — n: nb of rows

↳ Ex:

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 4 \end{bmatrix} \rightarrow 3A = \begin{bmatrix} 6 & 12 \\ 3 & 12 \end{bmatrix}$$

عدد الصفوف

$\det(A) = 4$

$$\det(3A) = 3^2 \det(A) = 9 \times 4 = 36$$

4. Adding a multiple of row to another

If matrix B result from A by adding a mutiple o[f] one row/column to another row/column then,

$\det(A) = \det(B)$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \rightarrow B = \begin{bmatrix} 1 & 2 \\ 5 & 8 \end{bmatrix} \quad 2R_1 + R_2 \rightarrow R_2$$

$\det(A) = \det(B) = -2$

5- Multiple of a row

If A is a square matrix with two proportional rows or columns, then $\det(A) = 0$

أي اذا كانت مصفوفة تحتوي على صف من مضاعفات صف آخر

↳ Ex: $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ $\quad R_2 = 2R_1$

$\Rightarrow \det(A) = 0$

6- Same rows

If in the matrix A two rows or columns are the same then, $\det(A) = 0$

↳ Ex: $A = \begin{pmatrix} \underline{1 \quad 2 \quad 3} \\ 5 \quad 6 \quad 7 \\ \underline{1 \quad 2 \quad 3} \end{pmatrix} \Rightarrow \det(A) = 0$

8. The matrix A is invertible if $\det(A) \neq 0$

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

↳ Ex:

$A = \begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix}$ $\quad B = \begin{pmatrix} 2 & 3 \\ 5 & 1 \end{pmatrix}$ $\quad$ Find if A and B are invertible

$\det(A) = 0 \longrightarrow$ does not invertible

$\det(B) = -13 \neq 0 \longrightarrow B$ is invertible

$$\det(B^{-1}) = \frac{1}{\det(B)} = -\frac{1}{13}$$

II) Finding determinant using row operations

Let $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 1 & 2 & 3 \\ 4 & 5 & 4 & 3 \\ 2 & 2 & -4 & 5 \end{pmatrix}$ Find det (A)

$R_2 - 5R_1 \to R_2$

$R_3 - 4R_1 \to R_3$

$R_4 - 2R_1 \to R_4$

$B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -9 & -13 & -17 \\ 0 & -3 & -8 & -13 \\ 0 & -2 & -10 & -3 \end{pmatrix}$

$$\det(B) = (1) \begin{vmatrix} -9 & -13 & -17 \\ -3 & -8 & -13 \\ -2 & -10 & -3 \end{vmatrix}$$

$$= -9 \begin{vmatrix} -8 & -13 \\ -10 & -3 \end{vmatrix} + 13 \begin{vmatrix} -3 & -13 \\ -2 & -3 \end{vmatrix} - 17 \begin{vmatrix} -3 & -8 \\ -2 & -10 \end{vmatrix}$$

$$= -9(-106) + 13(-17) - 17(14)$$

$$= 495$$

III) Finding Inverse using determinant

Let $A = \begin{pmatrix} 1 & 2 & 6 \\ 5 & 8 & 7 \\ 0 & 3 & 4 \end{pmatrix}$. Find the inverse of A if possible

1. Check if A is invertible:

$$\det(A) = 1\begin{pmatrix} 8 & 7 \\ 3 & 4 \end{pmatrix} - 5\begin{pmatrix} 2 & 6 \\ 3 & 4 \end{pmatrix}$$

$$= 1(11) - 5(-10)$$

$$= 61 \neq 0 \Rightarrow A \text{ is invertible}$$

2. Find the cofactor matrix:

$$Cof(A) \begin{bmatrix} \begin{vmatrix} 8 & 7 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 5 & 7 \\ 0 & 4 \end{vmatrix} & \begin{vmatrix} 5 & 8 \\ 0 & 3 \end{vmatrix} \\ -\begin{vmatrix} 2 & 6 \\ 3 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 6 \\ 0 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} \\ \begin{vmatrix} 2 & 6 \\ 8 & 7 \end{vmatrix} & -\begin{vmatrix} 1 & 6 \\ 5 & 7 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 5 & 8 \end{vmatrix} \end{bmatrix} \Rightarrow C \begin{bmatrix} 11 & -20 & 15 \\ 10 & 4 & -3 \\ -34 & 23 & -2 \end{bmatrix}$$

3. Find the transpose of the cofactor matrix (Adjoint matrix): $Adj(A) = [Cof(A)]^T$

$$Adj(A) = \begin{bmatrix} 11 & 10 & -34 \\ -20 & 4 & 23 \\ 15 & -3 & -2 \end{bmatrix}$$

4. Find the inverse using the formula:

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

$$= \frac{1}{61} \times \begin{bmatrix} 11 & 10 & -3 \\ -20 & 4 & 23 \\ 15 & -3 & -2 \end{bmatrix}$$